Some Fundamental Characteristics of a Discrete Geometry

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Among the fundamental principles which have been pursued by those researching the combinatorial hierarchy are: discreteness, finitism, and constructivism. When combined with primitive recursive operations for counting, labeling, and ordering, a rich variety of mathematical, logical, and physical structures can be obtained. In recent years, efforts to construct physical theories based upon the hierarchy have met some success. However, certain issues must, sooner or later, be addressed if such constructions are to be accepted by the scientific community-at-large.

Whenever a new theory is developed, its acceptance is predicated on the ability to communicate the spirit, the formal structure, and the utility of the theory. For the combinatorial hierarchy, that the theory be "communicatable" is an exceedingly difficult requirement to meet. Most of physics is understood in terms of centuries old geometric paradigms: the continuum, limits, infinities, distance functions (even distances), metrics, etc. How does one talk about distance in a finite world of discrete objects without the underlying continuum? What is a (physical) vector space in such a world? An event? A collision? Is it even possible to construct a discrete, finite model of the physical world without having such "old" language creep in, let alone be able to describe the model to a physicist steeped in a paradigm whose roots are at least as old as the Pythagorean irrational?

The most common form of constructivism is that which insists on the ability to provide a (algorithmic) construction of each object and theorem in the system. We refer to this as "weak constructivism". In the strong form of constructivism, each object and theorem of the system must be formulated along strict finitist and pure discretist lines. In this paper, we shall use "constructively", constructivist, and constructivism to refer to the principle of strong constructivism. The following principles (McGoveran's Principles) may be taken as guidelines in pursuing answers to the questions raised in the preceeding paragraph:

1) There is nothing in the knowable (or observable) Universe which can not be described constructively.

2) There is nothing which can be described constructively which (that known as) the physical Universe can not produce (in the combinatoric sense).

3) There is nothing which can be observed or known which can not be described constructively.

It is not enough to state such principles; one must demonstrate their utility. In particular, these principles deny the necessity, relevance, and even the meaningfulness of concepts such as infinity, infinitesimal, randomness, asynchrony, and continuity. At best, such concepts are useful only as placeholders in incomplete models, theories, specifications, or descriptions of a system. In most cases, randomness and asynchrony in particular are local descriptions of more global properties.

We believe it is possible to build a conceptual bridge which would allow the operational use of conventional terminology without implying (or assuming) the usual underlying geometric paradigm. First attempts at constructing such a bridge (and simultaneously bringing powerful tools to bear on the efforts at hand) have been made by developing a "discrete differential topology". This effort provides a means of refering to distances, functions, derivatives, etc. with most of the standard rules of use intact, but without violating strictures against appeal to non-constructivist entities such as the continuum, limits, or infinities.

In going from a discrete topology to a geometry, a number of interesting problems arise. So much that is taken for granted is not generally given meaning by construction: the geometric constants, the trigonometric functions, notions of direction, etc. It is interesting to note that most of the non-finite, non-discrete entities are embodied and related in one beautiful equation:

$$e^{i} = -1$$

Within this single relationship lies the assumptions that give rise to coordinate systems, the structures of a circle and a square, translational and rotational invariance, irrational and complex numbers, and, last but not least, the base of the natural logarithm. Even the notion that one could in any way raise a number to an arbitrary (perhaps non-integral) power is rooted in the geometric paradigm. Certainly complex numbers can be understood as ordered pairs of real numbers and these real numbers can be restricted to the positive integers for our purposes. And negative integers can be understood as the "dual" set of symbols obtained by "counting down" rather than "counting up". What of the other entities?

In what follows we construct a square and a circle, and derive a ratio which plays the role of pi. We begin by constructing the equivalent of a square, two-dimensional coordinate patch. The only elements allowed for construction are a finite (perhaps large) number of discrete elements (essentially indistinguishable mathematical objects), ordering relation operators, the ability to count, and the ability to label the objects through an operator.

Without reference to a distance function, a "square" can be defined having the following properties:

a) two-dimensionality
b) parallel sides
c) fixed center under interchange of the dimensional parameters

The following definitions will serve to provide the objects necessary for the constructions (more precise definitions can be found in "Getting Into Paradox"):

Def: Two objects are said to be INDISTINGUISHABLE if they are unlabeled.

Def: A SORT is an ensemble of indistiguishables with cardinality and without ordinality.

Def: A SET is a sort with ordinality (an ensemble is not a set).

Def: An ENUMERATION is a total ordering of a sort.

Def: An ORDERING OPERATOR is an operator which generates a partially or totally ordered labeling of an ensemble (note that we do not mean set). The cardinality of the labeled ensemble is fixed in advance as part of the definition of the specific operator. Thus a particular finite partially or totally ordered labeled ensemble defines an ordering operator and vice-versa. Note that the labeled ensembles produced are sets if and only if none of the labeled indistinguishables are twins.

Def: The DIMENSION of a sort is expressed as the number of mutually disjoint ordering operators on the sort.

Def: Two ordering operators are said to be INDEPENDENT if they are mutually disjoint in the sense that no more than one element of a chain produced by the first operator will all be also be in a chain produced by the second operator.

Def: By NEAREST NEIGHBOR of an element e in a chain ordered by operator p is meant any element n such that for a:a=p(e) or b:e=p(b); then for operator q mutually disjoint from operator p, n=q(e), n=q(a), n=q(b), e=q(n), a=q(n), or b=q(n).

The criteria for 2-dimensionality is satisfied by requiring two mutually disjoint ordering operators. The algorithm is as follows:

1. Select an element e_0 --- Figure 1.

2. Establish a chain ${}^{-1}x_0$ of length n with e_0 as the supremum, using the ordering operator p_x. --- Figure 2.

3. Establish a chain ${}^{1}x_0$ of length n with e_0 as the infimum, using the ordering operator p_x. ---- Figure 3.

4. Call the union of ${}^{-1}x_0$ and ${}^{1}x_0$; x_0. Require that x_0 is totally ordered. --- Figure 4.

5. For each element i of x_0, establish chains ${}^{-1}y_i$ and ${}^{1}y_i$ of length n under the ordering operator p_y with the selected element of x_0 as the supremum and infimum of the chain. Require that the y_i are disjoint as are the pairs (${}^{-1}y_i$, ${}^{1}y_i$). This is a unique labeling or total ordering requirement on the entire construction (i.e. there must exist an ordering operator q such that the elements of the entire construction are totally ordered. --- Figure 5.

6. Require that the nth elements of the y_i form chains x_i ordered by ordering operator p_{x_i}. --- Figure 6.

7. The resulting object satisfies the requirements: it is the discretum version of a 2-dimensional (square) coordinate patch. In particular, the 2-dimensionality of the construction is satisfied by the definition of mutually disjoint ordering operators: at most one element in a chain resulting from one operator will be found in a chain resulting from the other. For the given construction, at most two operators can be used: a third would result in a partial ordering instead of a total ordering of the elements of the construction and this would then represent an object which is not connected or in an object for which "multiple" elements are doubly labeled. Thus, the ordering operators "parameterize" the object.

We can now proceed to construct an object which behaves as a discretum version of the 2-sphere. A 2-sphere (again without reference to distance functions) has the following properties:

a) 2-dimensionality
b) every perimeter (boundary) element is like every other
c) fixed center for all "orientations"

The constructive algorithm is as follows:

1. Select a (square) coordinate patch with with center e_0 and all elements uniquely labeled. Call this patch P_0. Figure 7.

2. Constrain the possible ordering operators to those operators which produce chains of length n and which select for e_0 a nearest neighbor of e_0, then a nearest neighbor of this element, and so on. We refer to the operators which represent these selections as "radial permutations" of the coordinate patch. Figure 8.

3. Starting from e_0 construct a coordinate patch with new ordering operators which are radial permutations of coordinate patch. Figure 9.

4. Map the elements of this patch P_i to patch P_0 and eliminate any elements which do not have at least i labels, Figure 10.

5. Repeat this process for all pairs of allowed radial permutations. Figure 11.

The result is a discretum version of the circle, in that it has a fixed center (e_0) with radial symmetry (isomorphic to its radial permutations with identified center e_0). It has built in bounds on "precision". The relation between the number of "sides" of the polygon formed by a set of cardinality n and the number of permutations is fixed: it gives a measure of the "size" of the circle.

Given these two geometric objects it is possible to define a ratio which plays the role of the ratio of the area of the circle to the area of the square patch from which it was formed. This number is obtained by counting the number of elements contained in the circle and the number of elements contained in the square and forming the ratio.

A second ratio is obtained by counting the elements in the perimeter of the circle and forming the ratio with the length n of the chain x_0.

In general, these ratios will be functions of the length n of the chain x_0. Furthermore, the values of the ratios will not in general be those obtained under Euclidean geometry. However, if one insists on isotropy, homogeneity, and "density" (i.e. large n), it is easy to see that these values must be those obtained by the standard polygonal approximation to the circle. In particular, these ratios will be approximations to pi/4 and pi with the appropriate precision. These constructions and the results are closely related to numerical and statistical "approximation" methods as seen from within the traditional geometric paradigm. In fact, Archimedes came close to the construction used here (Measurement of the Circle). However, the definitions are completely constructive and general; matching the continuum definitions as desired.

It is a central point of this paper that a measure of the discrete cardinality and of the discrete topology of our observed Universe is given by the precision with which the ratios pi(area) and pi(lengths) are identical in value. That pi should be of cosmological significance is not surprising. Indeed, if the cardinality of the Universe is changing, then the two values of pi should be changing also. Furthermore, if the relevant discrete cardinality is related to a spatial volume, then as this region becomes smaller calculations involving pi can not be treated in a naive manner. Specifically, the multiple meanings of pi must be dissociated if the values are different (i.e., pi(areas)/pi(circumference) will not be 1) and the usual value can no longer be taken as a constant independent of spatial volume. Even more important, if the world is discrete and finite, and if the values of pi are not related to the spatial volume via a cardinality of the volume, it follows that the values of pi used in calculations relate only to the cardinality of the Universe. In other words, pi becomes a true Universal discrete topological constant and local physical properties are then immediately dependent on the global properties.

As noted above, we need only interpret complex numbers in the Hamiltonian sense in order to be consistent: namely, complex numbers are taken to be ordered pairs of numbers. This forces us to recognize two orderings at work for pairs of equations where as the imaginary notation obscures it. Complex numbers are a special case in that the additional ordering contains only a supremum and an infimum for the y_i. Thus, the commutation relation holds since commutation simply gives the trivial dual of the chain. In general, however, this will not be true where the constraints of disjoint operators apply and the number of ordered numbers is more than two. This fact is well known, having been discovered by Hamilton in trying to work out an algebra of quaternions, and having been generalized by Grassman. (Note that there are only three algebras which have an invertible vector product: those over the reals, those over the complex, and those over the quaternions (ordered 4-tuple). Also, algebras of all n-tuples greater than two are non-commutative.)

All of this leads us to consider a general property of discrete, finite spaces (now that we have a geometry we can call it a space instead of a topology), their commutivity. Assume for the purposes of illustration that a square coordinate patch is embedded in a flat space with continuous distance function. Because of the finiteness and the discreteness of the space, there can be no discrete correlate to the irrational distance of the diagonal, since according to the continuous distance function this will be square root of two times the length of a side. Thus, at best only an parallelogram law works for discrete distance functions on this space. Translated into the discrete version of Lie dragging, this means that the space has a torsion or is non-commutative. Geometrically one would say that the space is locally non-Euclidean. Alternatively, one could insist that there is no continuous distance function mappable to such a discrete space or that there is no discrete geometry. But this would be tantamount to a claim that all objects are non-local - i.e. infinitely extended. We prefer to conclude that discrete, finite geometries are not torsion-free.

FIGURE 1. e_0

```
-.-
```

FIGURE 2.
The sub-chain of length n=4, x_0^{-1} with e_0 as supremum.

```
o- -o- -o- -.-
```

FIGURE 3.
The sub-chain of length n=4, x_0^{1} with e_0 as infimum added.

x_0^{-1}

```
o- -o- -o- -.- -o- -o- -o
```

x_0^{1}

FIGURE 4. The chain of length n=7, x_0

x_0

```
o- -o- -o- -.- -o- -o- -o
```

FIGURE 5. The chains y_i of length n=7 added,

x_0

```
o    o    o    o    o    o    o
|    |    |    |    |    |    |
o    o    o    o    o    o    o
|    |    |    |    |    |    |
o    o    o    o    o    o    o
|    |    |    |    |    |    |
o- -o- -o- -.- -o- -o- -o
|    |    |    |    |    |    |
o    o    o    o    o    o    o
|    |    |    |    |    |    |
o    o    o    o    o    o    o
|    |    |    |    |    |    |
o    o    o    o    o    o    o
```

y_{-3} y_{-2} y_{-1} y_0 y_1 y_2 y_3

FIGURE 6. The chains x_i of length n=7 added.

```
x_3    o- -o- -o- -o- -o- -o- -o
       |   |   |   |   |   |   |
x_2    o- -o- -o- -o- -o- -o- -o
       |   |   |   |   |   |   |
x_1    o- -o- -o- -o- -o- -o- -o
       |   |   |   |   |   |   |
x_0    o- -o- -o- -.- -o- -o- -o
       |   |   |   |   |   |   |
x_-1   o- -o- -o- -o- -o- -o- -o
       |   |   |   |   |   |   |
x_-2   o- -o- -o- -o- -o- -o- -o
       |   |   |   |   |   |   |
x_-3   o- -o- -o- -o- -o- -o- -o

      y_-3 y_-2 y_-1 y_0 y_1 y_2 y_3
```

FIGURE 7. Select a patch, P_0.

```
x_2    o- -o- -o- -o- -o
       |   |   |   |   |
x_1    o- -o- -o- -o- -o
       |   |   |   |   |
x_0    o- -o- -.- -o- -o
       |   |   |   |   |
x_-1   o- -o- -o- -o- -o
       |   |   |   |   |
x_-2   o- -o- -o- -o- -o

      y_-2 y_-1 y_0 y_1 y_2
```

FIGURE 8. The nearest neighbors of e_0.

```
x                        o- -o- -o- -o- -o
 2                       |   |   |   |   |
x                        o- -*- -*- -*- -o
 1                       |   |   |   |   |
x                        o- -*- -.- -*- -o
 0                       |   |   |   |   |
x                        o- -*- -*- -*- -o
 -1                      |   |   |   |   |
x                        o- -o- -o- -o- -o
 -2

                         y   y   y   y   y
                          -2  -1  0   1   2
```

FIGURE 9. A new patch, P_i.

```
x                        o- -o- -o- -o- -o
 2                       |   |   |   |   |
x                        o- -o- -o- -o- -o
 1                       |   |   |   |   |
x                        o- -o- -.- -o- -o
 0                       |   |   |   |   |
x                        o- -o- -o- -o- -o
 -1                      |   |   |   |   |
x                        o- -o- -o- -o- -o
 -2

                         y   y   y   y   y
                          -2  -1  0   1   2
```

FIGURE 10. Mapping the new patch to the old.
* = new unmapped, # = old unmapped.

```
                                      *
                                     /
x              *    #- -o- -o- -o- -#
 2              \   | X | \ | \ | X |
x                   o- -o- -o- -o- -o
 1                  | / |   |   | / |
x                   o- -o- -.- -o- -o
 0                  | / |   |   | / |
x                   o- -o- -o- -o- -o
 -1                 | X | \ | \ | X | \
x                   #- -o- -o- -o- -#   *
 -2                   /
                    *

                    y   y   y   y   y
                     -2  -1  0   1   2
```

FIGURE 11. Elements remaining after all allowed radial permutations.

```
x                       o- -o- -o
 2                    / |   |   | \
x                   o- -o- -o- -o- -o
 1                  |   |   |   |   |
x                   o- -o- -.- -o- -o
 0                  |   |   |   |   |
x                   o- -o- -o- -o- -o
 -1                   \ |   |   | /
x                       o- -o- -o
 -2

                    y   y   y   y   y
                     -2  -1  0   1   2
```

CALCULATIONS:

```
A(square patch) = 16
A(polygon)      = 12
ratio = pi(areas)/4 =
 A(polygon)/A(square) =
        12/16 = 0.75
or pi(area) = 3.00
```

```
C(polygon) = 12
d(polygon) = 5
ratio = pi(lengths) =
 C(polygon)/d(polygon) =
          12/5 = 2.4
```

ADDENDUM:

Note that in the example, all other radial permutations cause the same elements of the construction to be deleted or else do not map the coordinant patch. The reader may demonstrate this for himself. Also note that a central element is a matter of technical convenience for the algorithms and may be circumvented.